New Zealand Mathematical Olympiad Committee

The Cauchy Functional Equation

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1 Introduction

Many functional equation problems can be reduced to solving a version of the Cauchy functional equation

$$f(x+y) = f(x) + f(y).$$

These notes explore families of solutions to this equation, and give some Olympiad problem applications.

2 Solutions on $\mathbb{Q}$

Let $f : \mathbb{R} \to \mathbb{R}$ be any function that satisfies the Cauchy equation. Clearly

$$f([k+1]x) = f(kx) + f(x);$$

hence by induction $f(nx) = nf(x)$ for all $n \in \mathbb{N}$.

We can extend this to all integers. First, $f(0) = f(0+0) = f(0) + f(0)$, so $f(0) = 0$. Then for any $n \in \mathbb{N}$

$$f(0) = f(nx + [-n]x) = nf(x) + f([-n]x),$$

so $f([-n]x) = [-n]f(x)$.

Finally, the result can be extended to all rationals, for if $m \in \mathbb{Z}$, $n \in \mathbb{N}$, then

$$mf(x) = f(mx) = f\left(n\frac{m}{n}x\right) = nf\left(\frac{m}{n}x\right),$$

and hence $f(\frac{m}{n}x) = \frac{m}{n}f(x)$.

In particular, we have proved that $f(x) = cx$ for all $x \in \mathbb{Q}$, where $c = f(1)$ is some fixed real.

3 'Nice' solutions on $\mathbb{R}$

Certainly all functions of the form $f(x) = cx$ (for some fixed c) are solutions to the Cauchy equation; let us call these linear functions the 'nice' solutions to the equation. Unfortunately, not all solutions to the Cauchy equation are 'nice'. However, all other solutions are very badly behaved indeed.

Theorem 1. *The only monotonic solutions to the Cauchy equation are the 'nice' solutions.*

Proof. Suppose f satisfies the Cauchy equation and is monotonic (that is, either increasing or decreasing). We know there is some $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in \mathbb{Q}$. We will prove that $f(x) = cx$ for all $x \in \mathbb{R}$.

If $c = 0$, then for all $x \in \mathbb{R}$, $\lfloor x \rfloor \le x \le \lceil x \rceil$, and $0 = f(\lfloor x \rfloor) = f(\lceil x \rceil)$, so $f(x) = 0 = cx$.

Otherwise, $c \neq 0$. Assume without loss of generality that $c > 0$ (the proof for $c < 0$ is almost identical). Suppose that for some $z \in \mathbb{R}$, $f(z) \neq cz$. Then $z \neq \frac{f(z)}{c}$. Between any two distinct real numbers there lies some rational; hence there exists $q \in \mathbb{Q}$ such that either $z < q < \frac{f(z)}{c}$ or $\frac{f(z)}{c} < q < z$.

If the former, $z < q$ but $f(q) = cq < f(z)$; if the latter, $q < z$ but $f(z) < cq = f(q)$. Either way, as f is monotonic, it follows that f must be decreasing. But as $0 < 1$ and $f(0) = 0 < c = f(1)$, this is a contradiction. □

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *continuous* if for all $x \in \mathbb{R}$ and any $\epsilon > 0$, we can find some $\delta > 0$ such that given any

$$x - \delta < y < x + \delta,$$

it is true that

$$f(x) - \epsilon < f(y) < f(x) + \epsilon.$$

Intuitively, a function is continuous if it can be drawn without lifting one's pen.

Theorem 2. *The only continuous solutions to the Cauchy equation are the 'nice' solutions.*

Proof. Suppose f satisfies the Cauchy equation. We know there is some $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in \mathbb{Q}$. We need to show that if for some $z \in \mathbb{R}$ $f(z) \neq cz$, then f is discontinuous.

Assume without loss of generality that $c \geq 0$ (the proof for $c \leq 0$ is almost identical). Let $\epsilon = |f(z) - cz|$; as $f(z) \neq cz$, $\epsilon > 0$. It suffices to show that for any positive δ, however small, there is some $z - \delta < y < z + \delta$ for which either

$$f(y) \leq f(z) - \epsilon, \quad \text{or} \quad f(y) \geq f(z) + \epsilon.$$

Indeed, suppose first that $f(z) > cz$. Between any two distinct reals lies some rational; hence we can choose some rational q for which $z - \delta < q < z$. For this q

$$\begin{aligned} f(q) &= (f(z) - cz) + f(q) - \epsilon \\ &= f(z) - cz + cq - \epsilon \\ &\leq f(z) - \epsilon. \end{aligned}$$

Otherwise $f(z) < cz$, and there is some $q \in \mathbb{Q}$ for which $z < q < z + \delta$. Then

$$\begin{aligned} f(q) &= -[cz - f(z)] + f(q) + \epsilon \\ &= f(z) - cz + cq + \epsilon \\ &\geq f(z) + \epsilon. \end{aligned}$$

□

Theorem 3. *Given any real interval, the only solutions to the Cauchy equation bounded on that interval are the 'nice' solutions.*

Proof. Let our interval be $[a, b]$. Suppose f satisfies the Cauchy equation and is bounded (that is, both bounded above and bounded below) on $[a, b]$. We know there is some $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in \mathbb{Q}$. We will prove that $f(x) = cx$ for all $x \in \mathbb{R}$.

Indeed, let x be any real. Between any two distinct reals lies some rational, so for each natural n there is some rational r_n such that $nx - b < r_n < nx - a$, and hence such that $nx - r_n \in [a, b]$.

$$\begin{aligned} |f(nx - r_n)| &= |nf(x) - cr_n| \\ &= |n(f(x) - cx) + c(nx - r_n)| \\ &\geq n|f(x) - cx| - |c(nx - r_n)| \\ |f(nx - r_n)| + |c(nx - r_n)| &\geq n|f(x) - cx|. \end{aligned}$$

Now $nx - r_n \in [a, b]$, so $c(nx - r_n)$ and $f(nx - r_n)$ are bounded as n increases. Hence $n|f(x) - cx|$ must be bounded too; it follows that $|f(x) - cx| = 0$, so $f(x) = cx$. □

4 Transforming the Cauchy equation

Many similar functional equations can be solved by an analogous process of

1. induction
2. extension to rationals
3. extension to reals,

or alternatively just by transforming the equation into the Cauchy equation.

Example 1. Find all monotonic functions $f : \mathbb{R} \to \mathbb{R}^+$ such that for all x and y,

$$f(x + y) = f(x)f(y).$$

Solution: Because $f(x) > 0$ for all x, we can define some new function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \log f(x)$. As the composition of monotonic functions, g is also monotonic. Rewriting the functional equation in terms of g, we find that for all x and y,

$$g(x + y) = g(x) + g(y).$$

That is, g satisfies the Cauchy functional equation; hence there is some $c \in \mathbb{R}$ such that for all real x, $g(x) = cx$ and $f(x) = e^{g(x)} = e^{cx}$. Equivalently, $f(x) = a^x$ for some fixed $a > 0$. It is easy to check that all such functions are solutions to the given problem. □

Example 2. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that for all x and y,

$$f(xy) = f(x)f(y).$$

Solution: Suppose that for some $k \neq 0$, $f(k) = 0$. Then for all real x, $f(x) = f(k)f(\frac{x}{k}) = 0f(\frac{x}{k}) = 0$. Clearly this zero function is a valid solution.

So let's look for other solutions to the equation. Clearly any such solution has $f(x) \neq 0$ for all $x \neq 0$. Also, for any $x \geq 0$, $f(x) = f(\sqrt{x})^2 \geq 0$. So for any $x > 0$ $f(x) > 0$. It follows that $g(t) = \log f(e^t)$ is well-defined for all $t \in \mathbb{R}$. As the composition of continuous functions, g is also continuous. Rewriting the functional equation in terms of g, we find that for all real s and t,

$$g(s + t) = g(s) + g(t).$$

That is, g satisfies the Cauchy functional equation; hence there is some $c \in \mathbb{R}$ such that for all real t, $g(t) = ct$, and for all $e^t = x > 0$,

$$f(x) = e^{g(\log x)} = e^{c \log x} = x^c.$$

As f is continuous, $f(x)$ must be bounded as x tends to zero; hence $c \geq 0$. Then as $f(-1)^2 = f(1) = 1^c = 1$, $f(-1) = \pm 1$, and for all negative x

$$f(x) = f(-1)f(-x) = f(-1)x^c.$$

If $c = 0$, $f(x) = 1$ for all $x > 0$, and $f(x) = f(-1)$ for all $x < 0$. So in this case, by continuity, $f(0) = 1$ and $f(-1) = 1$, yielding the (valid) solution $f(x) = 1$ to the original equation.

Otherwise by continuity $f(0) = \lim_{x \to 0} f(x) = 0$. These conditions yield two potential families of nonzero solutions:

$$f(x) = \begin{cases} x^c, & x \geq 0 \\ (-x)^c, & x < 0 \end{cases}$$

and

$$f(x) = \begin{cases} x^c, & x \geq 0 \\ -(-x)^c, & x < 0. \end{cases}$$

Again, it is easy to check that these are valid solutions. □

5 Problems

1. (IMO Shortlist 2003) Find all nondecreasing functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$, $f(1) = 1$, and for all real x and y such that $x < 1 < y$

$$f(x) + f(y) = f(x)f(y) + f(x + y - xy).$$

2. (IMO 1994) Find all functions $f : (-1, \infty) \to (-1, \infty)$ such that

   (i) $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$; and,

   (ii) $\frac{f(x)}{x}$ is strictly increasing on each of the intervals $(-1, 0)$ and $(0, \infty)$.

3. (IMO Shortlist 2003) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

   (i) $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all positive reals x, y, z; and,

   (ii) $f(x) < f(y)$ for all $1 < x < y$.

4. (IMO 2002) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y, z, t \in \mathbb{R}$,

$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz).$$

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