New Zealand Mathematical Olympiad Committee

Functional equations

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1 Composition of functions. Groups of functions.

Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be two functions. Then the *composition of f and g*, written $f \circ g$, is the function defined by $(f \circ g)(x) = f(g(x))$.

Note that the domain of $f \circ g$ consists of all x from the domain of g such that $g(x)$ belongs to the domain of f.

Exercise 1. Find the domains of $f \circ g$ and $g \circ f$, where

$$f(x) = 1 - x^2, \qquad g(x) = \sqrt{x}.$$

Exercise 2. For the functions

$$f(x) = \frac{x-2}{3x+4}, \qquad g(x) = \frac{2x-3}{5x-1}$$

find $f \circ g$ and $g \circ f$ and their domains.

Exercise 3. Find functions f, g such that $f \circ g \neq g \circ f$. Prove that for all functions f, g, h it is true that $(f \circ g) \circ h = f \circ (g \circ h)$.

Exercise 4. Let

$$f(x) = \frac{x}{\sqrt{1-x^2}}.$$

Find the n-fold composition $f^{(n)}(x) = f \circ f \circ f \circ \ldots \circ f(x)$.

We say that two functions f and g *commute* if $f \circ g = g \circ f$.

Exercise 5. Prove that for every function f the functions $f^{(n)}$ and$f^{(m)}$ commute for all m, n.

The function $e(x) = x$ has the property that $e \circ f = f \circ e = f$ for any other function f; e is sometimes called the *identity function.* A function g is called the inverse of f if $f \circ g = g \circ f = e$ or equivalently $f(g(x)) = g(f(x)) = x$ for all $x \in \mathbb{R}$. In this case we denote the inverse of f by f^{-1}.

A set G of functions is called a *group* if

1. $e \in G$;
2. If $f, g \in G$, then $f \circ g \in G$; and,
3. If $f \in G$, then $f^{-1} \in G$.

Exercise 6. Prove that the set G of linear functions $f(x) = ax + b$, $a \neq 0$, b arbitrary, is a group.

Exercise 7. Prove that the functions

$$g_1 = x, \qquad g_2 = 1 - x$$

form a group. Write down the “multiplication table” for this group.

Exercise 8. Prove that the functions

$$g_1 = x, \qquad g_2 = \frac{1}{1-x}, \qquad g_3 = \frac{x-1}{x}$$

form a group. Write down the “multiplication table” for this group.

Exercise 9. Prove that the functions

$$g_1 = x, \qquad g_2 = \frac{x-1}{x+1}, \qquad g_3 = -\frac{1}{x}, \qquad g_4 = \frac{x+1}{1-x}$$

form a group. Write down the "multiplication table" for this group.

We will use Exercise 7 to solve the functional equation

$$2f(1-x) + 1 = xf(x).$$

We substitite $1-x$ for x. As these two functions form a group we will again have an equation relating $f(x)$ and $f(1-x)$. Namely,

$$2f(x) + 1 = (1-x)f(1-x).$$

Now, making $f(1-x)$ the subject of the first equation and substituting it into the second, we get

$$2f(x) + 1 = (1-x)f(1-x) = (1-x)\cdot\frac{1}{2}(xf(x)-1)$$

and

$$f(x) = \frac{x-3}{x^2-x+4}.$$

This is a consequence of the original equation. Therefore we have to check that this function is indeed a solution. This can be easily done. I am not doing it here but it is a necessary part of the solution!

Exercise 10. Solve the functional equation $2xf(x) + f\left(\frac{1}{1-x}\right) = 2x$.

Exercise 11. Solve the functional equation $xf(x) + 2f\left(\frac{x-1}{x+1}\right) = 1$.

2 Cauchy's equation.

The equation

$$F(x+y) = F(x) + F(y)$$

is called the *Cauchy equation.* It is clear that any linear homogeneous function $F(x) = ax$ satisfies this equation.

Exercise 12. Let $F(x)$ be any function satisfying the Cauchy equation. Prove that for $a = F(1)$ we have $F(x) = ax$ for all rational numbers x.

Exercise 13. Prove that a continuous function satisfying the Cauchy equation must be a function $F(x) = ax$ for all real numbers x and $a = F(1)$.

But apart from linear functions there are many discontinuous solutions to the Cauchy equation.

In 1905 the German mathematician G. Hamel published in *Mathematische Annalen* the following result: he constructed a subset $G \subseteq \mathbb{R}$ with the property that every real number $x \in \mathbb{R}$ can be uniquely represented as a finite linear combination

$$x = n_1g_1 + n_2g_2 + \cdots + n_kg_k$$

of some $g_1, g_2, \ldots g_k \in G$ with nonzero integer coefficients $n_1, n_2, \ldots n_k$. The construction of this set G, which is known now as Hamel's basis, is quite tricky and cannot be given here.

Using this basis we can construct infinitely many solutions to the Cauchy equation, as follows. Let us put an arbitrary real number $a(g)$ in correspondence to an element $g \in G$ and define

$$F(x) = n_1a(g_1) + n_2a(g_2) + \cdots + n_ka(g_k).$$

To prove that $F(x)$ satisfies the Cauchy equation let us take also

$$y = m_1 g_1 + m_2 g_2 + \cdots + m_k g_k.$$

Without loss of generality we may assume that y is a linear combination of the same elements of G because we can always add some additional terms with zero coefficients. As

$$x + y = (n_1 + m_1)g_1 + (n_2 + m_2)g_2 + \cdots + (n_k + m_k)g_k$$

we have

$$F(x + y) = (n_1 + m_1)a(g_1) + (n_2 + m_2)a(g_2) + \cdots + (n_k + m_k)a(g_k) = F(x) + F(y),$$

so F satisfies the Cauchy equation, as claimed.

Discontinuous solutions are strangely behaved, and very hard to imagine. In fact, the graph of any discontinuous solution to the Cauchy equation is everywhere dense in the plane! As an example, we will prove one result about discontinous solutions.

Theorem 1. *Let $F(x)$ be a function satisfying the Cauchy equation. Suppose that $F(x)$ is bounded from above at least on one interval $(p, p + s)$, where s is a positive number. Then $F(x) = ax$ for all real numbers x, where $a = F(1)$.*

Proof. If the function $F(x)$ is bounded on the interval $(p, p + s)$, then it is bounded from above on the interval $(0, s)$. Indeed, for an arbitrary $x \in (0, s)$ there exists an upper bound M such that $F(p + x) < M$. Thus $F(p + x) = F(p) + F(x) < M$ and $F(x) < M - F(p)$ for an arbitrary $x \in (0, s)$. We may assume that s is rational. Let us consider now another function

$$G(x) = F(x) - \frac{F(s)}{s}x.$$

Clearly this function also satisfies the Cauchy equation and it is also bounded on the interval $(0, s)$. In addition we have $G(s) = 0$ and hence

$$G(x + s) = G(x) + g(s) = G(x)$$

which means that s is a period of $G(x)$! It follows now that the function $G(x)$ is bounded on the whole real line as every value of $G(x)$ is equal to some value of this function on the interval $(0, s)$. The function $G(x)$ must now be identically zero. Indeed, if $G(x_0) \neq 0$, then $G(nx_0) = nG(x_0)$, so G gets arbitrary large for large n which contradicts the fact that $G(x)$ is bounded. Thus $F(x) = \frac{F(s)}{s}x = F(1)x$. The theorem is proved. $\square$

Exercise 14. Analyse the Cauchy equation

$$F(x + y) = F(x)F(y)$$

by reducing it to the Cauchy equation by means of logarithmic transformation.

3 Problems

1. (Russia 1977)
   (a) For an arbitrary $\alpha \in \mathbb{R}$ find all polynomials of degree at most three, which commute with the polynomial $P(x) = x^2 - \alpha$.
   (b) Prove that there exists no more than one polynomial of degree n which commutes with the quadratic polynomial $P(x) = x^2 + px + q$.
   (c) Given a quadratic polynomial $P(x) = x^2 + px + q$, find all polynomials of degrees four and eight which commute with $P(x)$.
   (d) Prove that if two polynomials $Q(x)$ and $R(x)$ both commute with a fixed polynomial $P(x) = x^2+px+q$ of degree 2, then they commute, i.e. $Q \circ R = R \circ Q$.

(e) Prove that there exists a sequence of polynomials $P_1, P_2, \dots P_k, \dots$, such that the degree of P_k is equal to k, $P_2(x) = x^2 - 2$, and every two polynomials in this sequence commute. **Hint:** Use the *Chebycheff polynomials* $T_k(x)$ for which $T_k(\cos t) = \cos kt$.

2. (IMO 1973) Let $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for $j = 2, 3, \dots$. Prove that for any $n \in \mathbb{N}$ all roots of the equation $P_n(x) = x$ are real and distinct.

3. (IMO 1973) A nonempty set G of linear functions $f(x) = ax + b$, where a and b are real numbers, does not contain constant functions and has the following properties:

   (i) For arbitrary $f, g \in G$, we have $f \circ g \in G$, i.e. the set G is closed under composition of functions;

   (ii) If $f(x) = ax + b \in G$, then $f^{-1}(x) = \frac{1}{a}(x - b) \in G$; and,

   (iii) For every $f \in G$ there exists a number $a_f \in \mathbb{R}$ such that $f(a_f) = a_f$.

   Prove that there exists a number $a \in \mathbb{R}$ such that $f(a) = a$ for all $f \in G$.

4. (APMO 1989) Determine all functions from the reals to the reals for which

   (i) $f(x)$ is strictly increasing; and,

   (ii) $f(x) + g(x) = 2x$ for all real x, where $g(x)$ is the composition inverse function to $f(x)$.

   (Note: f and g are said to be *composition inverses*, if $f(g(x)) = x$ and $g(f(x)) = x$ for all real x.)

5. (IMO Shortlist 1997)

   (a) Do there exist functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ such that

   $$f(g(x)) = x^2 \qquad \text{and} \qquad g(f(x)) = x^3 \qquad \text{for all } x \in \mathbb{R}?$$

   (b) Do there exist functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ such that

   $$f(g(x)) = x^2 \qquad \text{and} \qquad g(f(x)) = x^4 \qquad \text{for all } x \in \mathbb{R}?$$

   (c) Find all functions f with the following properties: for all t,

   $$f(-t) = f(t), \quad f(t+1) = f(t) + 1, \quad f\left(\frac{1}{t}\right) = \frac{1}{t^2} f(t).$$

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