Stabbing all the k-convex hulls of points in a cyclic polytope using Kneser tranversals

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Introduction

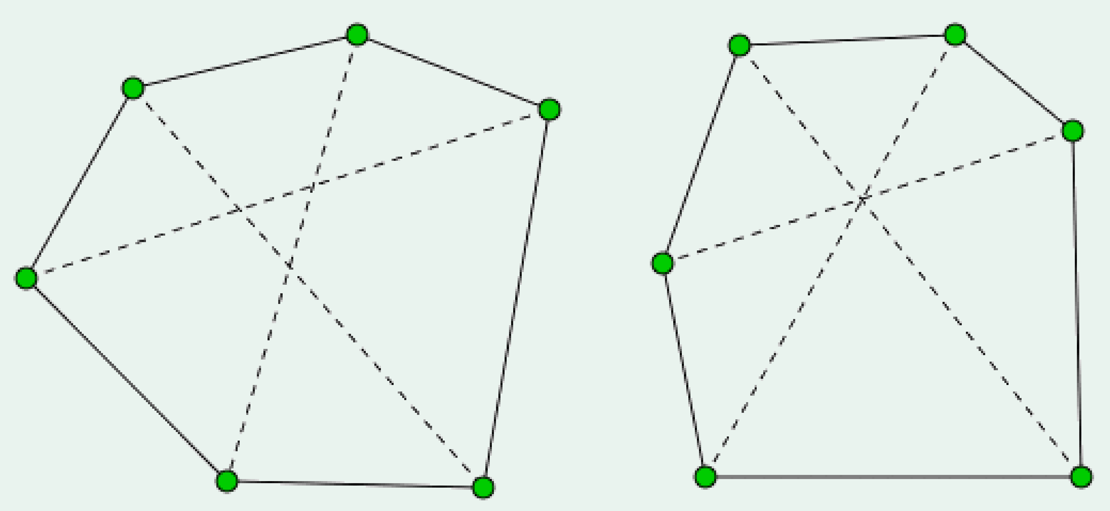
In 2010, Arocha, Bracho, Montejano and Ramírez-Alfonsín posed the following problem [1]. Let $k, d, \lambda \geq 1$ be integers with both $d, k \geq \lambda$.

What is the maximum number of points that we can have in $\mathbb{R}^d$ so that no matter how we choose them it is always possible to find a common $(d-\lambda)$-transversal plane to all the convex hulls of k-sets of the set of points? Call this number $m(k, d, \lambda)$.

As was pointed by the authors, and independently by B. Bukh, J. Matoušek and G. Nivash [2], this problem has connections to generalizations of two classical problems in combinatorial geometry: determining the chromatic number of Kneser hypergraphs and Rado's central point theorem. Arocha et al. also obtained upper and lower bounds for $m(k, d, \lambda)$ and stated a conjecture concerning its precise value.

This is a follow-up work on this problem. We introduce a discrete variant of the parameter that, unlike the original parameter, is invariant on the order type. This allows us to study the problem using the theory of oriented matroids. In particular, we can completely solve the problem asymptotically for when the points are the vertices of a cyclic polytope.

Example 1



Cyclic polytopes

The *moment curve* in $\mathbb{R}^d$ is defined parametrically as the map $\gamma : \mathbb{R} \to \mathbb{R}^d$, $t \mapsto (t, t^2, \ldots, t^d)$. A *cyclic polytope* is the convex hull of a finite set of points on the moment curve. The oriented matroids associated to cyclic polytopes on n vertices of dimension d are called *alternating oriented matroids*

References

[1] J.L. Arocha, J. Bracho, L. Montejano, J.L. Ramírez Alfonsín, Transversals to the convex hulls of all k-sets of discrete subsets of $\mathbb{R}^n$, J. Combin. Theory Ser. A 118 (2011), 197–207.

[2] B. Bukh, J. Matoušek, G. Nivasch, Stabbing simplices by points and flats, Discrete Comput. Geom. 43 (2010), 321–338.

[3] J. Chappelon, L. Martínez-Sandoval, L. Montejano, L.P. Montejano, J.R. Ramírez-Alfonsín, Complete Kneser Tranvsersals, arXiv preprint (2016).

For a complete list of references, see [3].

Acknowledgements

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The parameter m^* and some tools from convex geometry

The parameter m^* and two behaviours

The existence of a $(d-\lambda)$-tranversal to the convex hulls of the k-sets of a set of points is not an invariant on the order type (see Example 1). In this work we study the following parameter, which is an invariant on the order type.

$m^*(k, d, \lambda) \stackrel{\text{def}}{=}$ the maximum positive integer n such that every set of n points (not necessarily in general position) in $\mathbb{R}^d$ has a complete Kneser $(d-\lambda)$-transversal to the convex hulls of its k-sets.

The case $k = \lambda$ is easy to deal with, so from here on we will assume that $k \geq \lambda + 1$. It turns out that the function m^* has two different behaviors. The arguments for the case $\lambda - 1 \geq \lceil \frac{d}{2} \rceil$, are usually simpler than those for the case $\lambda - 1 < \lceil \frac{d}{2} \rceil$. For this reason, we define

$$\alpha(d, \lambda) = \frac{\lambda - 1}{\lceil \frac{d}{2} \rceil}$$

and we call $\alpha \geq 1$ the trivial range and $\alpha < 1$ the non-trivial range. In this poster we only state results for the non-trivial range.

Tools from convex geometry to study m^*

Let d be a positive integer. Consider $d+2$ points $v_1, v_2, \ldots, v_{d+2}$ in general position in $\mathbb{R}^d$. Radon's theorem states that there exists a unique partition $\{1, 2, \ldots, d+2\} = A \cup B$ such that

$$\operatorname{Conv}\left(\bigcup_{i \in A} v_i\right) \cap \operatorname{Conv}\left(\bigcup_{i \in B} v_i\right) \neq \emptyset.$$

Moreover, Radon's theorem states that this intersection is a unique point in the interior of each convex hull.

We use the following proposition is a generalization of the well-known Carathéodory's theorem.

Proposition 1. *Let d and λ be positive integers with $d \geq \lambda$ and let S and T be two disjoint sets of points in general position in $\mathbb{R}^d$ such that $|S| \geq \lambda + 1$ and $|T| = d - \lambda + 1$. Then the following two statements are equivalent:*

- $\operatorname{Conv}(S) \cap \operatorname{aff}(T) \neq 0$,
- $\operatorname{Conv}(S') \cap \operatorname{aff}(T) \neq 0$ *for a subset $S' \subseteq S$ such that $|S'| = \lambda + 1$.*

We also use the following criteria to detect the intersection of an affine hull and a convex hull generated by some sets of points.

Proposition 2. *Let S and T be two disjoint and non-empty sets of points in $\mathbb{R}^d$ such that $|S| + |T| = d + 2$ and $S \cup T$ is in general position. Then $\operatorname{Conv}(S) \cap \operatorname{aff}(T) \neq \emptyset$ if and only if all the points of S are in the same set in the Radon partition of $S \cup T$.*

Notice that these results prove that m^* is an invariant of the order type.

The cyclic polytope and main results

In order to give bounds for m^*, we study the case in which the points are the vertices of a cyclic polytope. For this we introduce the following function:

$\zeta(k, d, \lambda) \stackrel{\text{def}}{=}$ the maximum number of vertices that the cyclic polytope in $\mathbb{R}^d$ can have, so that it has an complete Kneser $(d-\lambda)$-transversal to the convex hulls of its k-sets of vertices.

We were able to determine the value of $\zeta(k, d, \lambda)$ asymptotically in k.

Theorem 3. *In the non trivial-range we have that*

$$\lim_{k \to \infty} \frac{\zeta(k, d, \lambda)}{k} = 2 - \alpha(d, \lambda).$$

To prove this result we proved the bounds $z(k, d, \lambda) \leq \zeta(k, d, \lambda) \leq Z(k, d, \lambda)$ where

$$z(k, d, \lambda) \stackrel{\text{def}}{=} (d - \lambda + 1) + \max_{\substack{j \in \{\lambda+1, \ldots, d-\lambda+2\} \\ j+\lambda \text{ is odd}}} \left(\left\lfloor \frac{k-1}{\beta(\lambda, j)} \right\rfloor \cdot j + (k-1)_{\operatorname{mod} \beta(\lambda, j)} \right)$$

$$Z(k, d, \lambda) \stackrel{\text{def}}{=} (d - \lambda + 1) + \lfloor (2 - \alpha(d, \lambda))(k-1) \rfloor$$

These bounds follow from a careful study of the alternating oriented matroids, from the introduction of some combinatorial tools and from the convex geometry tools stated above.

Using Theorem 3, and an additional argument for a lower bound, we provide the following result.

Theorem 4. *In the non-trivial range we have that*

$$(d - \lambda + 1) + k \leq m^*(k, d, \lambda) \leq \zeta(k, d, \lambda) \leq Z(k, d, \lambda).$$