A Hall-type theorem for points in general position

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Geometry 30 Years Later

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Hall's theorem

Let $F = \{S_1, \ldots, S_m\}$ be a family of finite subsets of a common ground set E. A system of distinct representatives is an $m-$element subset $\{x_1, x_2, \ldots, x_m\}$ of E such that $x_i \in S_i$ for all $1 \leq i \leq m$.

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Theorem (Hall's theorem, 1935)

The family F has a system of distinct representatives if and only if for every subset $I \subset \{1, 2, \ldots, m\}$ we have

$$\left| \bigcup_{i \in I} S_i \right| \geq |I|.$$

Hall-type theorems

Let $F = \{S_1, \ldots, S_m\}$ be a family of subsets on a common ground set E. A system of _____ representatives is an $m-$element subset $\{x_1, x_2, \ldots, x_m\}$ of E with the property _____ such that $x_i \in S_i$ for all $1 \leq i \leq m$.

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Theorem (Hall-type theorem)

The family F has a system of _____ representatives if (and only if) for every subset $I \subset \{1, 2, \ldots, m\}$ we have

$$\alpha\left(\bigcup_{i \in I} S_i\right) \geq f(|I|).$$

Here α is an integer valued function related to the conclusion we want to obtain.

Hall-type theorem for hypergraphs

Let $F = \{H_1, \ldots, H_m\}$ be a family of hypergraphs on a common vertex set V. A system of disjoint representatives is an $m-$element subset $\{E_1, E_2, \ldots, E_m\}$ of pariwise disjoint edges such that $E_i \in H_i$ for all $1 \leq i \leq m$.

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Theorem (Aharoni-Haxel, 2000)

The family F has a system of disjoint representatives if for every subset $I \subset \{1, 2, \ldots, m\}$ there exists a matching M_I in $\cup_{i \in I} H_i$ which needs at least $|I|$ disjoint edges from $\cup_{i \in I} H_i$ to be pinned.

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The proof introduced topological techniques to the area.

General position in $\mathbb{R}^d$

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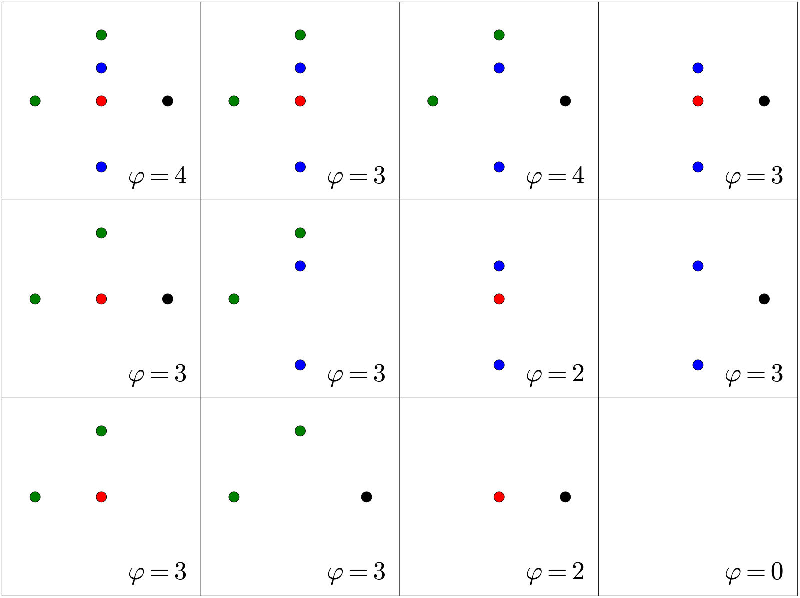
- For a set X of points in $\mathbb{R}^d$ let $\varphi(X)$ denote the maximal size of a subset of X in general position.
- Let $F = \{X_1, \ldots, X_m\}$ be a family of finite sets in $\mathbb{R}^d$. A system of general position representatives is a $m-$element subset $\{x_1, x_2, \ldots, x_m\}$ in general position such that $x_i \in X_i$ for all $1 \leq i \leq m$.

Example


$\varphi = 4$
$\varphi = 3$
$\varphi = 4$
$\varphi = 3$
$\varphi = 3$
$\varphi = 3$
$\varphi = 2$
$\varphi = 3$
$\varphi = 3$
$\varphi = 3$
$\varphi = 2$
$\varphi = 0$

Hall-type theorem for general position

Theorem (A. Holmsen, L.M-S., L. Montejano, 2015)

For every integer $d \geq 1$ there exists a function $f_d : \mathbb{N} \to \mathbb{N}$ with $f_d(k)$ in $O(k^d)$ such that the following holds.
Let $F = \{X_1 \ldots, X_m\}$ be a family of finite sets in $\mathbb{R}^d$. If

$$\varphi\left(\bigcup_{i\in I} X_i\right) \geq f_d(|I|)$$

for every non-empty subset $I \subset \{1, 2, \ldots, m\}$, then F has a system of general position representatives.

Sketch of the proof

We study a simplicial complex K

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We call K the general position complex.

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We study a simplicial complex K

$$\begin{aligned} \text{Vertices of } K: &\quad \cup F \\ \text{Faces of } K: &\quad \text{Sets in general position} \end{aligned}$$

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$$\begin{array}{ccc} \bigcup_{i \in I} X_i & \longrightarrow & \text{Induced sub complex of } K \\ \text{System of g.p.r.} & \longleftarrow & \text{Colorful simplex} \end{array}$$

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- In particular, we can understand the connectivity of K by verifying a local condition on L. The following lemma can be deduced from Björner's version of the Nerve Theorem (2003).

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- In particular, we can understand the connectivity of K by verifying a local condition on L. The following lemma can be deduced from Björner's version of the Nerve Theorem (2003).

Lemma (A. Holmsen, L.M-S., L. Montejano)

Let L be a simplicial complex of dimension d and let k be a non-negative integer. If L is $(2k+2)$-star, then its d-completion $\Delta_d(K)$ is k-connected.

Geometric Hall-type theorems

- Therefore, for any $I \subset \{1, 2, \ldots, m\}$ and $k = |I| - 2$, if L_I is the subcomplex of L induced by $\cup_{i \in I} X_i$ we have:

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Remark: Using an easier pidgeon-hole argument we can get a function $f_d(k)$ in $O(k^{d+1})$. The topological technique allows us to get a function in $O(k^d)$. In some sense this is asymptotically correct.

A lower bound

For k and d positive integers we define

$$C_d(k) = \begin{cases} k & \text{if } k \leq d+1 \\ \binom{k-1}{d} & \text{if } k \geq d+2. \end{cases}$$

Proposition

Let d be a positive integer. Let m be an integer $m \geq d+2$. There exists an example of a family $F = \{X_1, \ldots, X_m\}$ of finite sets in $\mathbb{R}^d$ without a system of general position representatives and for which

$$\varphi\left(\bigcup_{i \in I} X_i\right) \geq C_d(|I|)$$

for every non-empty subset $I \subseteq \{1, 2, \ldots, m\}$.

Open problems

Computation

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Further work

- Which other geometric properties have a Hall-type theorem?

Thank you!

Thank you for your attention!